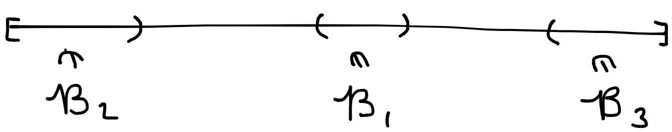
The order topology

Let X be an ordered set (w/ $<$ and $\leq$). We can define a topology on X (using the order relation) called the "order topology."

Def: Let $\mathcal{B}_1 = \{(a,b) \mid a < b\}$

If X has a smallest element a_0, $\mathcal{B}_2 = \{[a_0, b) \mid b \in X\}$, otherwise $\mathcal{B}_2 = \emptyset$.

If X has a largest element b_0, $\mathcal{B}_3 = \{(a, b_0] \mid a \in X\}$, otherwise $\mathcal{B}_3 = \emptyset$.



Define $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. Then $\mathcal{B}$ is a basis for a topology on X called the <u>order topology</u>.

Need to check $\mathcal{B}$ is in fact a basis:

If $x \in X$, then if x is the smallest element, it is in an interval of type 2. If it's the largest element, it is in an interval of type 3. Otherwise, $\exists\ a, b \in X$ s.t. $a < x < b$, so $x \in (a,b)$.

The intersection of any two elements of $\mathcal{B}$ is also in $\mathcal{B}$ or empty (to check, analyze cases).

Example: Consider $\mathbb{R}$ w/ the standard order. Then since $\mathbb{R}$ has no largest or smallest element, a basis for the order topology on

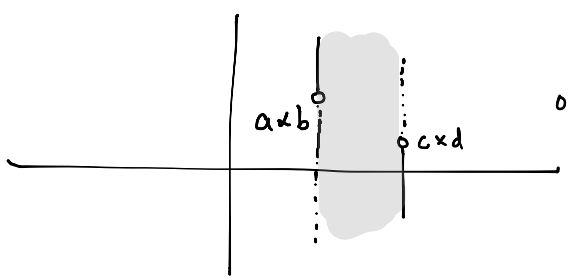
$\mathbb{R}$ is $\mathcal{B}=\{(a,b) \mid a<b\}$. Thus, the order topology on $\mathbb{R}$ is equal to the standard topology.

Example: Consider $\mathbb{R}\times\mathbb{R}$ w/ the dictionary order. In order to avoid confusion, write the element $(a,b)\in\mathbb{R}\times\mathbb{R}$ as $a\times b$.

Recall that $a\times b < c\times d$ if $a<c$ or $a=c$ and $b<d$.

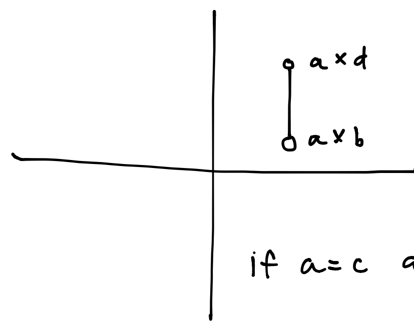
Thus, a basis for the order topology is the collection of open intervals of the form $(a\times b, c\times d)$ where $a\times b < c\times d$.

So basis elements look like this:



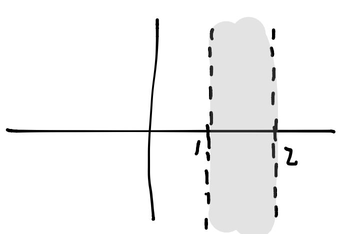
if $a<c$

or



if $a=c$ and $b<d$

Consider the set $U=(1,2)\times\mathbb{R}$ in $\mathbb{R}\times\mathbb{R}$:



This is not a basis element!

Suppose $U=(a\times b, c\times d)$.

If $1<a<2$, then $a\times(b-1) < a\times b$, which is a contradiction.

Thus, $a\leq 1$ so $a\times b < a\times(b+1)\notin U$, again a contradiction. Thus, U is not a basis element.

Is U open?

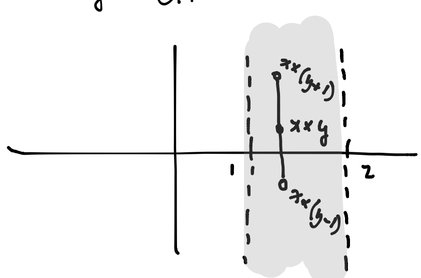
Let $\mathcal{C} = \{ (a \times b, c \times d) \mid a \times b < c \times d, \ 1 < a \leq c < 2 \}$.

Notice that $\mathcal{C}$ is a collection of basis elements.

Define $V = \bigcup_{S \in \mathcal{C}} S$. Clearly $V \subseteq U$, since each $S \subseteq U$.

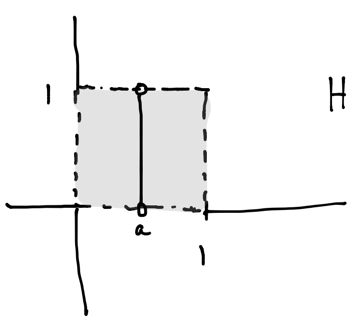
Want to show $V = U$.

Let $x \times y \in U$. Then $1 < x < 2$, so $x \times y \in (x \times (y-1), x \times (y+1)) \in \mathcal{C}$



Thus, $x \times y \in V$, so $V = U$. Thus U is a union of basis elements, so it's open.

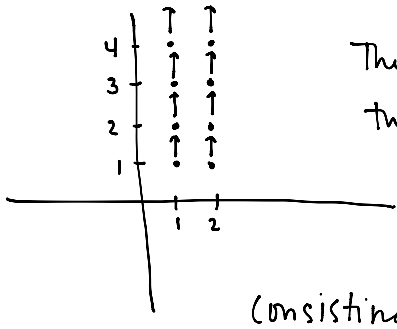
Now let $W = (0,1) \times (0,1)$. Again, this is certainly not a basis element. (Do you see why?).



However, W is a union of open intervals of the form $(a \times 0, a \times 1)$, where $0 < a < 1$.
Thus, W is open!

Example: The order topology on $\mathbb{Z}$ is the discrete topology (on pset).

Example: Consider $X = \{1, 2\} \times \mathbb{Z}_+$ w/ the dictionary order.



The order topology on X is <u>not</u> the discrete topology!

Consider the set $U = \{2 \times 1\}$ consisting of a single element.

If U is open, then there must be a basis element in U that contains 2×1. Thus, U itself must be a basis element.

Suppose $U = (a \times b, c \times d)$. Then $a \times b < 2 \times 1$, so $a = 1$. Thus,

$1 \times b < 1 \times (b+1) < 2 \times 1$, which is a contradiction since $1 \times (b+1) \notin U$.

Thus, U is not open, so the order topology is not the discrete topology in this case.